NORTH-HOLLAND

A Remark on the Eigenvalue Distribution of the Preconditioning Collocation Scheme with $\mathbb{P}_3$ Finite Element Stiffness Matrix*

Sang Dong Kim

Department of Mathematics
Teachers College
KyungPook National University
Taegu, 702-701, Korea

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ABSTRACT

For the preconditioning collocation scheme using a Hermite (or interpolatory) cubic spline basis for a 1D elliptic model problem, we prove that the multiplicity of the minimal eigenvalue 1 is the half the order of the preconditioned collocation matrix.

1. INTRODUCTION

Let I be the unit interval $[0, 1]$, and consider a 1D model elliptic problem

$$-u''(x) = f(x) \tag{1.1a}$$

with the mixed boundary condition

$$u(0) = 0 \quad \text{and} \quad u'(1) = 0. \tag{1.1b}$$

The cubic interpolatory spline basis to solve a second order separable elliptic partial differential equation by the preconditioning collocation method was introduced in [3] and [4], where the $\mathbb{P}_1$ (or $\mathbb{P}_3$) finite element stiffness matrix was used as preconditioner. Let $\widetilde{B}$ and $\widetilde{\beta}$ be the symmetrized collocation matrix and finite element discretization of (1.1) based on a Hermite cubic (or interpolatory) spline basis and collocation at the Gaussian points

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$\{\xi_i\}_{i=1}^{2N}$. The symmetrized preconditioning collocation scheme corresponding to (1.1) is (see Section 3 for the definitions of the matrices)

$$\widetilde{\beta}^{-1}\widetilde{B}U := \widetilde{\beta}^{-1}E^tWBU = \widetilde{\beta}^{-1}E^tWF.$$

It is well known that the investigation of the eigenvalues of the preconditioning collocation scheme is important for the successful application of the conjugate gradient method. Even though there are full descriptions in [4] of the uniform boundedness of the eigenvalues for the preconditioning collocation scheme $\widetilde{\beta}^{-1}\widetilde{B}$ by the finite element method using the prescribed basis functions for S_h, there is no detailed discussion of the distribution of the eigenvalues of this matrix. In particular, we report that, when the $\mathbb{P}_3$ finite element stiffness matrix is used as preconditioner for the model problem (1.1), the multiplicity of the minimal eigenvalue 1 of $\widetilde{\beta}^{-1}\widetilde{B}$ is half the order of the collocation matrix.

2. PRELIMINARY

Let $N > 1$ be an integer, and set $h = 1/N$. The *knots* are the points

$$x_k := kh, \qquad k = 0, 1, 2, \ldots, N,$$

and

$$I_k := (x_{k-1}, x_k), \qquad k = 1, 2, \ldots, N.$$

Let S_h be the space of Hermite cubic splines defined on I with knots x_k. That is,

$$S_h := \{u(x) \in C^1[0,1],\ u|_{I_k} \in \mathbb{P}_3,\ u(0) = u'(1) = 0\}.$$

The basis functions $v_j(x), s_j(x)$ $(j = 0, 1, \ldots, N)$ for S_h are given by the translation and dilation of the functions $v(x), s(x)$ given by

$$v(x) := \begin{cases} v^l(x) = (x+1)^2(1-2x), & -1 \le x \le 0, \\ v^r(x) = (x-1)^2(2x+1), & 0 \le x \le 1, \end{cases}$$

$$s(x) := \begin{cases} s^l(x) = (x+1)^2(1-2x), & -1 \le x \le 0, \\ v^r(x) = (x-1)^2(2x+1), & 0 \le x \le 1. \end{cases}$$

Then we can write $v_j(x), s_j(x)$, where $j = 0, 1, \ldots, N$, as

$$v_j(x) := \begin{cases} vs\left(\dfrac{x - x_j}{h}\right), & x_{j-1} \le x \le x_{j+1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$s_j(x) := \begin{cases} hs\left(\dfrac{x - x_j}{h}\right), & x_{j-1} \le x \le x_{j+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{\phi_i\}_{i=1}^{2N}$ be the Hermite cubic spline basis such that

$$\phi_{2i-1} = s_{i-1}, \quad \phi_{2j} = v_j, \qquad j = 1, 2, \ldots, N.$$

Let

$$c_1 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right), \qquad c_2 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right).$$

Then the collocation points $\{\xi_i\}_{i=1}^{2N}$ are given by

$$\xi_{2i-1} = x_{i-1} + hc_1, \quad \xi_{2i} = x_{i-1} + hc_2, \qquad i = 1, 2, \ldots, N.$$

Let $\{\psi_i\}_{i=1}^{2N}$ be the cubic interpolatory spline basis for the space S_h (see [3, 4]) such that

$$\psi_i(\xi_j) = \delta_{ij}, \qquad \text{where} \quad \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Finally we define, for $u \in S_h$,

$$\langle -u'', u\rangle_N = -\frac{h}{2} \sum_{i=1}^{2N} u''(\xi_i) u(\xi_i).$$

3. COUNTING EIGENVALUES

In this section we will present main results. In [2], there is a very interesting result which suggests the finite element preconditioner for the collocation method.

LEMMA 1 [2]. *For $u \in S_h$, we have*

$$\|u'\|_0^2 \le \langle -u'', u\rangle_N \le \tfrac{5}{3}\|u'\|_0^2.$$

Proof. See [2]. ∎

Now let $u = \sum_{r=1}^{2N} u_i \phi_i(x) \in S_h$. Then the collocation version of (1.1) will be

$$\sum_{r=1}^{2N} -u_i \phi_r''(\xi_l) = f(\xi_l), \qquad l = 1, 2, \ldots, 2N. \tag{3.1}$$

Let us define the $2N \times 2N$ matrices

$$B(j,r) = -\phi_r''(\xi_j), \qquad E(j,r) = \phi_r(\xi_j), \qquad W(j,r) = \mathrm{diag}(\omega_j).$$

The matrix representation of (3.1) is

$$BU := F, \qquad \text{where} \quad F = (f(\xi_1), \ldots, f(\xi_{2N}))^t, \quad U = (u_1, \ldots, u_{2N})^t.$$

Let

$$\widetilde{B} := E^t W B. \tag{3.2}$$

The preconditioning collocation scheme corresponding to (1.1) is then

$$\widetilde{\beta}^{-1}\widetilde{B}U = \widetilde{\beta}^{-1}E^t WF. \tag{3.3}$$

Note that when we use the interpolatory cubic spline basis, E is the identity matrix. Now we need

LEMMA 2. *For $u \in S_h$, we have*

$$\langle -u'', u\rangle_N = (u', u')_{L_2} + c\sum_{i=1}^{N} \left|u_i^{(3)}\right|^2 h^5 \tag{3.4a}$$

In matrix language, we have

$$\widetilde{B}_{l,k} = \widetilde{\beta}_{l,k} + c d_{l,k}^* \tag{3.4b}$$

where

$$\widetilde{\beta}_{l,k} := (\phi_l', \phi_k')_{L_2}, \qquad d_{l,k}^* = \sum_{i=1}^{N} \phi_{l,i}^{(3)} \phi_{k,i}^{(3)} h^5,$$

where $u_i^{(3)}$ is the constant third derivative of u_i on I_i, and c is a positive constant independent of N.

Proof. Use the similar proof of Lemma 3.1 in [1]. ∎

This lemma allows us to count eigenvalues for the preconditioning collocation linear system, which is

$$\beta^{-1}\widetilde{B}U = \beta^{-1}E^t WF.$$

By direct computation (see [5]) we have the following block tridiagonal matrix D_{2N}^*:

$$D_{2N}^* := \left(d_{l,k}^*\right) = \begin{pmatrix} D_1 & U_1 & 0 & \cdots & \cdots & \cdots & 0 \\ L_1 & D_3 & U & 0 & \cdots & \cdots & 0 \\ 0 & L & D_3 & U & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & L & D_3 & U & 0 \\ 0 & \cdots & \cdots & 0 & L & D_3 & U_2 \\ 0 & \cdots & \cdots & \cdots & 0 & L_2 & D_2 \end{pmatrix}_{2N\times 2N}, \tag{3.5}$$

where

$$D_1 = \begin{pmatrix} h & -2 & h \\ -2 & 8/h & 0 \\ h & 0 & 2h \end{pmatrix}, \qquad D_3 = \begin{pmatrix} 8/h & 0 \\ 0 & 2h \end{pmatrix},$$

$$D_2 = \begin{pmatrix} 8/h & 0 & -4/h \\ 0 & 2h & -2 \\ -4/h & -2 & 4/h \end{pmatrix},$$

$$U_1 = \begin{pmatrix} 0 & -4/h & -2 \\ 0 & 2 & h \end{pmatrix}^t, \qquad U = \begin{pmatrix} -4/h & 2 \\ -2 & h \end{pmatrix},$$

$$U_2 = \begin{pmatrix} -4/h & 2 & 0 \\ -2 & h & 0 \end{pmatrix},$$

and

$$L = U^t, \qquad L_1 = U_1^t, \qquad L_2 = U_2^t.$$

Let I_n be the identity matrix of dimension n, and M_k^n be an elementary lower triangular matrix of order n and index k (see [6]). Note that this M_k^n is nonsingular. Using elementary lower triangular matrices, we can prove the following lemma, which is the essential key to the main theorem.

LEMMA 3. *The nullity of D_{2N}^* is N for $N \geq 3$.*

Proof. The proof will be completed by performing Gaussian elimination for the matrix D_{2N}^* inductively on the order of D_{2N}^*. First consider the case $N = 3$. Now construct the elementary lower triangular matrices M_1^6, M_2^6, and M_3^6 from D_6^* to perform Gaussian elimination. Then by simple calculations we have

$$M_4^6 M_2^6 M_1^6 D_6^* = \begin{pmatrix} h & -2 & h & 0 & 9 & 0 \\ 0 & 4/h & 2 & -4/h & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4/h & 2 & -4/h \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.6}$$

Since each elementary matrix is nonsingular, we can see the nullity of D_6^* is 3. Now for the induction hypothesis assume that we have performed Gaussian elimination on D^* of order $2(N-1)$. As a result we have the set of the elementary lower triangular matrices M_i^{2N-2} of order $2N-2$ and index $i = 1, 2, 4, 6, \ldots, 2N-6, 2N-4$, and then the resulting matrix is

$$D_{2N-2}^{**} = M_{2N-4}^{2N-2} \overbrace{M_{2N-6}^{2N-2} \times \cdots \times M_4^{2N-2} M_2^{2N-2} M_1^{2N-2}}^{F \overset{\text{def}}{=}} D_{2N-2}^*.$$

Now for the $2N \times 2N$ matrix D_{2N}^* let

$$M_{2N-2}^{2N} = \begin{pmatrix} I_{2N-2} & 0 \\ D & I_2 \end{pmatrix}, \qquad M_{2N-4}^{2N} = \begin{pmatrix} M_{2N-4}^{2N-2} & 0 \\ C & I_2 \end{pmatrix},$$

where

$$D = \begin{pmatrix} 0 & \cdots & 0 & -h/2 \\ 0 & \cdots & 0 & 1 \end{pmatrix}_{2\times(2N-2)},$$

$$C = \begin{pmatrix} 0 & \cdots & 0 & -h/2 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}_{2\times(2N-2)}.$$

For $i = 1, 2, 4, \ldots, 2N-6$ let the elementary lower triangular matrix of order $2N$ and index i be of the form

$$M_i^{2N} = \begin{pmatrix} M_i^{2N-2} & 0 \\ 0 & I_2 \end{pmatrix}.$$

Let us decompose D^* of order $2N$ as

$$D_{2N}^* = \begin{pmatrix} A_{2N-2} & B \\ B^t & E \end{pmatrix},$$

where

$$A_{2N-2} = D^*_{(2N-2)} + \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 4/h \end{pmatrix}_{(2N-2)\times(2N-2)},$$

$$B = \begin{pmatrix} 0 & \cdots & 0 & 2 & h & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & -4/h \end{pmatrix}^t, \qquad E = \begin{pmatrix} 2h & -2 \\ -2 & 4/h \end{pmatrix}.$$

Note that by computation we have

$$F = \begin{pmatrix} * & 0 \\ * & I_3 \end{pmatrix}, \qquad M^{2N-2}_{2N-4} F = \begin{pmatrix} * & \cdots & * & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & * & 0 & 0 & 0 \\ * & \cdots & * & -h/2 & 1 & 0 \\ * & \cdots & * & 1 & 0 & 1 \end{pmatrix}$$

of order $2N-2$. Then we have the following matrix of similar form to (3.6):

$$\begin{aligned} & M^{2N}_{2N-2} M^{2N}_{2N-4} \times \cdots \times M^{2N}_2 M^{2N}_1 D^*_{2N} \\ &= \begin{pmatrix} I_{2N-2} & 0 \\ D & I_2 \end{pmatrix} \begin{pmatrix} M^{2N-2}_{2N-4} & 0 \\ C & I_2 \end{pmatrix} \begin{pmatrix} FA_{2N-2} & FB \\ B^t & E \end{pmatrix} \\ &= \begin{pmatrix} M^{2N-2}_{2N-4} F A_{2N-2} & M^{2N-2}_{2N-4} FB \\ DM^{2N-2}_{2N-4} FA_{2N-2} + CFA_{2N-2} + B^t & DM^{2N-2}_{2N-4} FB + CFB + E \end{pmatrix} \\ &= \begin{pmatrix} U_1 & U_2 & U_3 & & & 0 \\ & U & U_4 & U_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & U & U_4 & U_3 \\ & & & & U & U_4 \\ 0 & & & & & U_N \end{pmatrix}, \end{aligned} \tag{3.7}$$

where

$$U_1 = \begin{pmatrix} h & -2 \\ 0 & 4/h \end{pmatrix}_{2\times2}, \qquad U_2 = \begin{pmatrix} h & 0 \\ 2 & -4/h \end{pmatrix}_{2\times2}, \qquad U_3 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}_{2\times2},$$

$$U = \begin{pmatrix} 0 & 0 \\ 0 & 4/h \end{pmatrix}_{2\times2}, \qquad U_4 = \begin{pmatrix} 0 & 0 \\ 2 & -4/h \end{pmatrix}_{2\times2}, \qquad U_N = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{2\times2}.$$

The last equality in (3.7) is easily checked by using the induction hypothesis and direct computations. Since each M_i^{2N} is invertible, we have the conclusion from (3.7). ∎

Because of Lemma 3, we can now prove the main result.

THEOREM 1. *The preconditioned symmetrized collocation matrix* $\widetilde{\beta}^{-1}\widetilde{B}$ *generated by the Hermite cubic spline basis has a minimal eigenvalue* 1 *whose multiplicity is half the order of the collocation matrix.*

Proof. By (3.4) we have

$$\widetilde{\beta}^{-1}\widetilde{B} = I + c\widetilde{\beta}^{-1}D^*.$$

Since $\widetilde{\beta}$ is nonsingular, Lemmas 3 and 1 yield the conclusion. ∎

REMARK. We have the same result for the interpolatory cubic spline by applying the change of the basis to Theorem 1.

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